Structural Contradictions

Cindy Eisner
IBM Haifa Research Laboratory

Joint work with Dana Fisman

Structural contradiction

- A SERE[‡] not satisfiable due to its structure

$$p \cap (p \cdot p)$$

- Contrast with logical contradiction
  - Propositional formula that is not satisfiable

$$p \wedge \neg p$$

‡ Semi-extended regular expression

The structure

- Logical contradiction
  - Exists valid proposition of same structure

$p \wedge \neg p$ → **$true \wedge \neg false$**

- Structural contradiction
  - No SERE of same structure is satisfiable

$p \cap (p \cdot p)$ → **$true \cap (true \cdot true)$**

More on structure

- For a propositional formula
  - One kind of contradiction (logical)

$$p \wedge \neg p$$

- For a SERE
  - Two kinds (structural and non-structural)

$$p \cap (p \cdot p)$$

$$(p \cdot p) \cap (p \cdot \neg p)$$

∩ is not the only source

- The fusion operator ∘
  - A kind of overlapping concatenation
  - $\mathcal{L}(r_1 \circ r_2) = \{u\ell v \mid u\ell \in \mathcal{L}(r_1) \text{ and } \ell v \in \mathcal{L}(r_2)\}$

<a><b><cd><e><f>

$\in \mathcal{L}((\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}) \circ (\mathbf{d} \cdot \mathbf{e} \cdot \mathbf{f}))$

<a> is a letter such that atomic proposition **a** and only atomic proposition **a** holds

In social theory

- **Structural contradiction**
  - Contradiction that can only be resolved by dismantling existing structure of power*
- According to Marxist theory
  - **Capitalism** is a structural contradiction
  - Though Karl Marx viewed bourgeois society as more of a process than a structure‡

* John Croft, history forum, amazon.com
‡ Andy Blunden of marxists.org, personal communication

Back to temporal logic

Why do we care?

Due to lack of time, we'll skip the truncated semantics and show only the issue of r! vs. r

See paper for all the gory details on the truncated semantics

- Similar issue in **truncated semantics**

Outline

- **The semantics of SERE formulas**
- What breaks wrt structural contradictions
- Some fixes that do not work
- One fix that does
- Good stuff about the fix
- Structural contradictions not created equal
- Conclusion

Semantics, intuition

There exists a prefix in $\mathscr{L}$(r):

Concatenation:

<a><b><b><c> $\in \mathscr{L}((a \cdot b) \cdot (b \cdot c))$

Overlapping concatenation:

<a><b><c> $\in \mathscr{L}((a \cdot b) \circ (b \cdot c))$

<a><b>$^{\omega} \models (a \cdot b^{*} \cdot c)$

Note similarity to weak until (W)

r!	r	r ↦ φ

The ⊤,⊥ approach

- Augment alphabet with two special letters
  - Anything holds on ⊤ (even false)
  - Nothing holds on ⊥ (even true)
- Define $\overline{w}$ to be the word obtained by
  - Replacing every ⊤ with ⊥ and vice versa
- LTL operators
  - Standard, except that negation uses $\overline{w}$

$$\mathbf{w} \models_{\top\bot} \neg\varphi \iff \overline{\mathbf{w}} \not\models_{\top\bot} \varphi$$

Semantics of ⊤,⊥ approach

- SERE formulas

$$w \models_{\top\bot} r! \iff \exists j < |w| \text{ s.t. } w^{0..j} \in \mathcal{L}(r)$$

$$w \models_{\top\bot} r \iff \forall j < |w|,\ w^{0..j}\top^{\omega} \models_{\top\bot} r!$$

$$w \models_{\top\bot} r \mapsto \varphi \iff$$
$$\forall j < |w| \text{ s.t. } \overline{w}^{0..j} \in \mathcal{L}(r),\ w^{j..} \models_{\top\bot} \varphi$$

Outline

- The semantics of SERE formulas
- **What breaks wrt structural contradictions**
- Some fixes that do not work
- One fix that does
- Good stuff about the fix
- Structural contradictions not created equal
- Conclusion

What breaks

- Let $\varphi = (a \cdot b^* \cdot \text{false})$
  $\varphi' = (a \cdot b^* \cdot (c \cap (c \cdot c)))$
- Hopefully, it is easy to see that

$\langle a\rangle\langle b\rangle\langle b\rangle\langle b\rangle \models_{\top\bot} \varphi$

$\langle a\rangle\langle b\rangle\langle b\rangle\langle b\rangle \not\models_{\top\bot} \varphi'$

- T "catches" the false, but not $(c \cap (c \cdot c)))$

$$w \models_{\top\bot} r \iff \forall j < |w|,\ w^{0..j}\top^{\omega} \models_{\top\bot} r!$$

Outline

- The semantics of SERE formulas
- What breaks wrt structural contradictions
- **Some fixes that do not work**
- One fix th... es

Why am I showing you these?

To show that it's not so easy to get it right.

(We tried and discarded these fixes.)

Bad fix 1: “flexible letters”

- Define
  - $\mathcal{L}(\lambda) = T^*$
  - $\mathcal{L}(b) = \{\ell \text{ s.t. } \ell \Vdash b\} \cdot T^*$
- Now

One T for false

<a><b><b><b> $\vDash_{T\bot}$ $(a \cdot b^* \cdot \text{false})$

<a><b><b><b> $\vDash_{T\bot}$ $(a \cdot b^* \cdot (c \cap (c \cdot c))$

Two T's for c

And two for $(c \cdot c)$

So what's w…

We've changed the semantics!

- Let $r_1 = (a \cdot b \cdot c)$, $r_2 = (d \cdot e \cdot f)$, $\varphi = r_1 \circ r_2$

Original semantics (one T per atom)	Flexible letter approach
$\langle a\rangle\langle b\rangle\langle c\rangle \not\models_{T\perp} \varphi$	$\langle a\rangle\langle b\rangle\langle c\rangle \models_{T\perp} \varphi$

$\langle a\rangle\langle b\rangle\langle c\rangle$ "matches" r_1,
but $\langle c\rangle TT$ does not "match" r_2

we can use $\langle a\rangle\langle b\rangle\langle c\rangle T$ to "match" r_1, and TTT to "match" r_2

Bad fix 2: weak language

"Too short"

"Too long"

- … SERE, include … t" or "too long"

$\mathcal{L}_{weak}(\lambda) = \varepsilon$

$\mathcal{L}_{weak}(b) = \varepsilon \cup \{\ell \text{ s.t. } \ell \Vdash b\}$

$\mathcal{L}_{weak}(r_1 \cdot r_2) = \mathcal{L}_{weak}(r_1) \cup (\mathcal{L}(r_1) \cdot \mathcal{L}_{weak}(r_2))$

$\mathcal{L}_{weak}(r_1 \circ r_2) = \mathcal{L}_{weak}(r_1) \cup (\mathcal{L}(r_1) \circ \mathcal{L}_{weak}(r_2))$

$\mathcal{L}_{weak}(r^*) = (\mathcal{L}(r)^* \cdot \mathcal{L}_{weak}(r)) \cup (\mathcal{L}(r))^\omega$

$\mathcal{L}_{weak}(r_1 \cup r_2) = \mathcal{L}_{weak}(r_1) \cup \mathcal{L}_{weak}(r_2)$

$\mathcal{L}_{weak}(r_1 \cap r_2) = \mathcal{L}_{weak}(r_1) \cap \mathcal{L}_{weak}(r_2)$

Bad fix 2: weak language

- Define

$$\mathbf{w} \models \mathbf{r} \iff \exists j < |w| \text{ s.t. } w^{0..j} \in \mathcal{L}(r) \text{ or } r \in \mathcal{L}_{weak}(r)$$

Now what's wrong?

Another problem:
Should not have mixed the
"too short" & "too long" cases

Reason is related to
truncated semantics;
see paper for details

(one

but <c>TT does not "match" r_2

<a><b><c> $\in \mathcal{L}_{weak}(r_1 \circ r_2)$

Outline

- The semantics of SERE formulas
- What breaks wrt structural contradictions
- Some fixes that do not work
- **One fix that does**
- Good stuff about the fix
- Structural contradictions not created equal
- Conclusion

Three languages of a SERE

- $\mathcal{L}(r)$ — as before
- $\mathcal{F}(r)$
  - Finite **proper** prefixes of words in $\mathcal{L}(r)$
  - Except: no opinion on satisfiability
  - $\mathcal{F}(a \cdot b \cdot c) = \mathcal{F}(a \cdot b \cdot false) = \mathcal{F}(a \cdot b \cdot (c \cap (c \cdot c)))$
- $\mathcal{I}(r)$
  - Infinite words in which we "get stuck" in a *

Formal…

No ω here, only words that are "too short"

$\mathcal{F}(\lambda) = \emptyset$

$\mathcal{F}(b) = \varepsilon$

Proper prefix!

$\mathcal{F}(r_1 \cdot r_2) = \mathcal{F}(r_1) \cup (\mathcal{L}(r_1) \cdot \mathcal{F}(r_2))$

$\mathcal{F}(r_1 \circ r_2) = \mathcal{F}(r_1) \cup (\mathcal{L}(r_1) \circ \mathcal{F}(r_2))$

$\mathcal{F}(r^*) = \mathcal{L}(r)^* \cdot \mathcal{F}(r)$

$\mathcal{F}(r_1 \cup r_2) = \mathcal{F}(r_1) \cup \mathcal{F}(r_2)$

$\mathcal{F}(r_1 \cap r_2) = \mathcal{F}(r_1) \cap \mathcal{F}(r_2)$

Formally

Both empty – you can only get stuck in a *

Here's the ω

$\mathcal{I}(\lambda) = \emptyset$

$\mathcal{I}(b) = \emptyset$

$\mathcal{I}(r_1 \cdot r_2) = \mathcal{I}(r_1) \cup (\mathcal{L}(r_1) \cdot \mathcal{I}(r_2))$

$\mathcal{I}(r_1 \circ r_2) = \mathcal{I}(r_1) \cup (\mathcal{L}(r_1) \circ \mathcal{I}(r_2))$

$\mathcal{I}(r^*) = (\mathcal{L}(r)^* \cdot \mathcal{I}(r)) \cup (\mathcal{L}(r))^\omega$

$\mathcal{I}(r_1 \cup r_2) = \mathcal{I}(r_1) \cup \mathcal{I}(r_2)$

$\mathcal{I}(r_1 \cap r_2) = \mathcal{I}(r_1) \cap \mathcal{I}(r_2)$

Semantics

- w ⊨ r ⇔ ∃j>0 s.t. $w^{0..j} \in \mathcal{L}(r)$ or $w \in \mathcal{I}(r) \cup \mathcal{F}(r) \cup \{\varepsilon\}$

By using proper prefixes, we avoid the problems of the bad fixes previously shown

…ate $\mathcal{I}(\cdot)$ from $\mathcal{F}(\cdot)$ …o play until we …ated semantics … …r details

…ve …to …ls on the empty word

- …
- w…

Outline

- The semantics of SERE formulas
- What breaks wrt structural contradictions
- Some fixes that do not work
- One fix that does
- **Good stuff about the fix**
- Structural contradictions not created equal
- Conclusion

Good stuff about the fix

- Logical vs. structural treated consistently
- Preserves existing semantics
  - In the absence of structural contradictions
- Preserves properties of truncated semantics
  - ε models everything weakly, nothing strongly
  - Strength relation theorem
  - Prefix extension theorem
- Complexity is not affected

Outline

- The semantics of SERE formulas
- What breaks wrt structural contradictions
- Some fixes that do not work
- One fix that does
- Good stuff about the fix
- **Structural contradictions not created equal**
- Conclusion

Back to political theory

- "We hold these truths to be self-evident, that **all men are created equal**"
  - Th. Jefferson, *Declaration of Independence*, 1776

- Surprisingly, this **does not hold** for **structural contradictions**

Not created equal

- Let $\varphi_1 = true \cap (true \cdot true)$
  $\varphi_2 = (true \cdot true) \cap (true \cdot true \cdot true)$
- Then on a word of length one
  - φ_1 does not hold
  - φ_2 does
- Note element of temporality
  - $\varphi_2 = true \cdot (true \cap (true \cdot true))$
    $= X\ \varphi_1$

Order of a formula

- Length of longest word w s.t. $w \models^{-} \varphi$
- Formulas of order 0:
  - false
  - true $\cap$ (true · true)
- Formulas of order 1:
  - X false
  - (true · true) $\cap$ (true · true · true)

Still not created equal!

- Let $\varphi_1 = (p \cdot p) \cap (p \cdot p \cdot p)$
  $\varphi_2 = (\mathrm{true} \cdot \mathrm{true}) \cap (\mathrm{true} \cdot \mathrm{true} \cdot \mathrm{true})$
- Then on w = <>
  - φ_1 does not hold
  - φ_2 does

One letter, no propositions hold

- Although both are of order 1

Logical vs. structural

false Ξ true ∩ (true · true)

- And in general

X^{n-1} false Ξ $true^{n}$ ∩ ($true^{n}$ · $true^{+}$)

- But what about **(λ ∘ true)** ?
  - Doesn't it have a logical equivalent?
  - Yes, but there are subtleties
    - See paper for details

Outline

- The semantics of SERE formulas
- What breaks wrt structural contradictions
- Some fixes that do not work
- One fix that does
- Good stuff about the fix
- Structural contradictions not created equal
- **Conclusion**

Future work:

find a topological characterization that works for $\cap$ and $\circ$ in new semantics

(Previously:

topological characterization proving that without $\cap$ and $\circ$, the relation between r! and r is the same as between U and W)